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On Group Averaging for $SO(n,1)$

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Abstract

The technique known as group averaging provides powerful machinery for the study of constrained systems. However, it is likely to be well defined only in a limited set of cases. Here, we investigate the possibility of using a ‘renormalized’ group averaging in certain models. The results of our study may indicate a general connection between superselection sectors and the rate of divergence of the group averaging integral.

I. INTRODUCTION

We are interested here in what one might call the practical implementation of the Dirac Quantization procedure [1] for constrained systems. Recall that the Dirac approach involves introducing the constraints as operators on some space and then taking only those states which are annihilated by the constraints to be ‘physical.’ These physical states are then made into a (physical) Hilbert space. Recall also that the Dirac procedure and the closely related BRST approach [2] are the favored methods for addressing quantum gauge systems.

A number of variants of the Dirac method have been discussed, including geometric quantization [3], reduced phase space methods [4,5], coherent state quantization [6], algebraic quantization [7], and refined algebraic quantization [8,9] (in which we include the work of [10–12]). It is refined algebraic quantization (RAQ) in particular that we will study here. RAQ has been shown to have a certain generality [9] and has the useful property that the classical reality conditions of an observable algebra are implemented as hermiticity relations of the operators on the physical Hilbert space *without* first constructing the quantum observables explicitly [8]. However, refined algebraic quantization becomes much more powerful when a technique known as ‘group averaging’ can be applied. Group averaging uses the integral

$$\int_G \langle \phi_1 | U(g) | \phi_2 \rangle dg \tag{1.1}$$

over the gauge group G to define the physical Hilbert space. Here dg is what one might call the ‘symmetric’ Haar measure on G [13]. Once a space of states (Φ) has been found for which this procedure converges, group averaging gives an *algorithm* for the implementation of RAQ. When group averaging converges sufficiently strongly¹, this algorithm gives the *unique* implementation of RAQ [13]. In particular, group averaging provides the unique Hilbert space representation (with a unique inner product) of the algebra of observables which is compatible with RAQ. Convergent group averaging also gives an algorithm for construction of a complete set of observables [13]. The convergence of group averaging is typical in mini-superspace settings, in which it has been used to construct physically meaningful observables [12] as well as to study the semi-classical limit [14,15] and, in particular, the instanton approximation [15]. Although the influence of the choice of domain Φ is not fully understood, we see that the case where group averaging converges is under fair control.

However, it will often happen that group averaging fails to converge on some interesting domain. As described in [13], the fact that convergent group averaging ensures a unique representation (compatible with RAQ) of the algebra of observables shows that group averaging *must* in fact diverge in the presence of any superselection rules. However, as was described in [8], one can sometimes construct a ‘renormalized’ group averaging operation, even when group averaging does not properly converge. Ref. [8] successfully used this idea

¹At least for locally compact (i.e., finite dimensional or non-field theoretic) gauge groups.

in the context of the loop approach [7,16] to quantum gravity to construct a Hilbert space of states which are invariant under the group of diffeomorphisms of a spacelike surface Σ . Our goal here is to construct further examples of successful ‘renormalized group averaging’ as a potential aid to its future general study.

Below, we consider as gauge groups the components $SO_c(n, 1)$ of $SO(n, 1)$ which are connected to the identity. As discussed in [13], group averaging is guaranteed to converge on the regular representation, in which $SO_c(n, 1)$ acts on $L^2(SO_c(n, 1))$. However, the most familiar representation of $SO_c(n, 1)$ is given by its action on $n + 1$ dimensional Minkowski space $M^{n,1}$. We consider here the associated representations of $SO_c(n, 1)$ on $L^2(M^{n,1})$. Section II studies the convergence of group averaging for various sectors. We find that group averaging converges for states corresponding to smooth functions f on $M^{n,1}$ when the closure of the support of f lies inside the light cone. For $n > 1$, group averaging does not converge for states whose support extends outside the light cone. However, we show that a certain ‘renormalization’ of the group averaging scheme does lead to a well-defined physical inner product. We then show in section III that this satisfies the detailed requirements of RAQ. In fact, we find a two parameter family of such physical Hilbert spaces. One parameter is a trivial overall normalization, but the other stems from a superselection rule between physical states associated with the interior of the light cone and those associated with the exterior. Further implications of our results are discussed in section IV. We will not review the details of group averaging and refined algebraic quantization here. Instead, we refer the reader to [8,9,13], whose notation we follow.

II. GROUP AVERAGING

Consider the group $G = SO_c(n, 1)$ acting on $L^2(M^{n,1}, d^n x)$. The infinitesimal action of the group is defined by the generators of the Lie Algebra,

$$J_{\mu\nu} = \eta_{\mu\alpha} x^\alpha \frac{\partial}{\partial x^\nu} - \eta_{\nu\alpha} x^\alpha \frac{\partial}{\partial x^\mu} , \quad (2.1)$$

whose exponentiation gives the unitary action $U(g)$ of the group. The generators $J_{\mu\nu}$ also define the constraints of the theory. Thus, physical states satisfy

$$J_{\mu\nu} |\psi\rangle_{\text{phys}} = 0 . \quad (2.2)$$

Since there are no such normalizable states, RAQ redefines this condition to be

$$\langle \psi |_{\text{phys}} J_{\mu\nu} = 0 . \quad (2.3)$$

We are interested in the convergence of the associated group averaging integral (1.1) on some domain Φ . If it converges, or if it can be renormalized in a useful way, it will define a map (known as the ‘rigging map’) from Φ into the space of physical states. Below, we study this issue by first finding a useful parameterization of the Haar measure on $SO_c(n, 1)$ in subsection A. We then perform explicit calculations of the group averaging integral in subsection B. In subsection C we present the final form of the resulting (candidate) rigging map. The proof that this is indeed a rigging map will be given in the section III.

A. The Haar Measure

Let us first find a parameterization of $SO_c(n, 1)$ and compute its Haar measure. Any element g in $G = SO_c(n, 1)$ is a product of a boost and a rotation. In general this is called the ‘‘Cartan decomposition’’ [17]. In our case, choosing some x^0 time coordinate in Minkowski space, we write $g = h_0 k_0$ for k_0 in the $SO_c(n)$ subgroup K of G that preserves the x^0 axis and h_0 a symmetric positive definite matrix (a pure boost). In general, such an h_0 can be written as $h_0 = k_1 b(\lambda) k_1^{-1}$ for a rotation $k_1 \in K$ and $b(\lambda)$ a boost (with boost parameter λ) in the x^0, x^1 plane. For our purposes, it is convenient to write $k = k_1^{-1} k_0$ and $h = k_1 b(\lambda)$ so that we have

$$g = hk, k \in K, \quad \text{and} \quad h \in H_+^n. \quad (2.4)$$

Note that H_+^n may be identified with the (right) coset space $SO_c(n, 1)/K$. It will be useful to represent this space as the upper sheet of the Hyperboloid

$$-(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = -1 \quad (2.5)$$

by mapping h to the image of the x^0 axis under h .

A generic element of H_+^n can be written

$$h = k_{n-1}(\theta_{n-1}) k_{n-2}(\theta_{n-2}) \cdots k_1(\theta_1) b(\lambda), \quad (2.6)$$

where k_m is a rotation in the plane (x^m, x^{m+1}) and $b(\lambda)$ is a hyperbolic rotation in (x^0, x^1) . Here $0 < \lambda < \infty$, $0 \leq \theta_i < \pi$ for $i = 1, \dots, n-2$ and $0 \leq \theta_{n-1} < 2\pi$. In terms of standard Minkowski coordinates and the identification of H_+^n with the upper sheet of the hyperboloid (2.5), the parameterization (2.6) is

$$\begin{aligned} x^0 &= \cosh \lambda \\ x^1 &= \sinh \lambda \cos \theta_1 \\ x^2 &= \sinh \lambda \sin \theta_1 \cos \theta_2 \\ &\vdots \\ x^n &= \sinh \lambda \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1} \end{aligned}$$

Now, the standard measure $d^{n+1}x$ on the region within the future light cone $x^0 > 0$, $x \cdot x < 0$ in $n+1$ Minkowski space is invariant under $SO_c(n, 1)$. Let s denote the timelike separation of a point x inside this light cone from the origin: $s^2 = -x \cdot x$. Writing the measure $d^{n+1}x$ as $s^n ds dh$ leads to an $SO_c(n, 1)$ -invariant measure dh for H_+^n given by

$$dh = \sinh^{n-1} \lambda \sin^{n-2} \theta_1 \cdots \sin \theta_{n-2} d\lambda d\theta_1 \cdots d\theta_{n-1}, \quad (2.7)$$

where $d\lambda$ and $d\theta_i$ are the usual Lebesgue measures on the appropriate intervals. Consider then the measure $dg(hk) = dh(h) dk(k)$ on $SO_c(n, 1)$, where $dk(k)$ denotes the Haar measure

on K . For any $g \in G$, we may write $gh = h_1 k_1$ for $h_1 \in H_+^n$ and $k_1 \in K$. In particular, h_1 is such that it takes the x^0 axis to the same line in $M^{n,1}$ as gh does. Thus, g acts as an $SO_c(n, 1)$ transformation on H_+^n and $dh(h_1) = dh(h)$. Since $dk(k_1 k) = dk(k)$, we have $dg(ghk) = dh(h_1)dk(k_1 k) = dg(g)$ and

$$dg = dh \, dk \quad (2.8)$$

is a Haar measure on G . For more details on the procedure to compute Haar measures for different parameterizations, see [18].

B. The averaging procedure

We wish to study the integral

$$\int_{g \in G} \langle \phi_1 | U(g) | \phi_2 \rangle dg , \quad (2.9)$$

where ϕ_1 and ϕ_2 lie in some domain $\Phi \subset \mathcal{H}_{\text{aux}}$. It is natural to take Φ to be a subspace of smooth functions of compact support. Thus, we proceed by introducing the distributional states $|x\rangle$ for $x \in M^{n,1}$ satisfying $\langle x_1 | x_2 \rangle = \delta^{n+1}(x_1, x_2)$. We will then study

$$I := \int_{g \in G} \langle x_1 | U(g) | x_2 \rangle dg , \quad (2.10)$$

treating this expression as a distribution in both x_1 and x_2 .

The expression (2.10) can be written as follows,

$$\begin{aligned} \int dg \langle x_1 | U(g) | x_2 \rangle &= \frac{1}{V_{SO(n)}} \int dk \int dg \langle x_1 | U(kg) | x_2 \rangle \\ &= \frac{1}{V_{SO(n)}} \int dk \, dh \, dk' \langle x_1 | U(khk') | x_2 \rangle , \end{aligned} \quad (2.11)$$

where $k, k' \in K$, $h \in H_+^n$ and $V_{SO(n)} = \int dk$ is the volume of $SO(n)$.

However, any element of h can be written as in (2.6). Thus, using the $SO(n)$ translation invariance of dk , equation (2.11) takes the form,

$$I = \frac{V_{S_{n-1}}}{V_{SO(n)}} \int dk \, dk' \, d\lambda \, \sinh^{n-1} \lambda \, \langle x_1 | U(k) U(b(\lambda)) U(k') | x_2 \rangle , \quad (2.12)$$

where $V_{S_{n-1}} = \frac{\pi^{n/2}}{\Gamma(n/2)}$ is the volume of the $(n-1)$ -sphere. Below, we write $U(b(\lambda))$ as $B(\lambda)$ to make the distinction clear between this boost and the rotations $U(k)$.

To evaluate the integral in (2.12) it is useful to introduce two complete sets of states, and to rewrite (2.12) as

$$\int_{k, k' \in K} dk dk' \, d\lambda d^{n+1}x \, d^{n+1}x' \langle x_1 | U(k) | x \rangle \langle x | B(\lambda) | x' \rangle \langle x' | U(k') | x_2 \rangle . \quad (2.13)$$

Averaging over the compact group $SO(n)$ is straightforward, and up to a constant factor yields

$$\int_K dk \langle x|U(k)|x' \rangle = \frac{1}{r^{n-2}} \delta(t, t') \delta(r^2, r'^2), \quad (2.14)$$

where $r^2 = \sum_{i>0} x^i x^i$. This may be seen from the fact that, if we assign each coordinate (t, x^i) dimensions of length, the matrix elements $\langle x|U(k)|x' \rangle$ have dimensions of $(\text{length})^{-(n+1)}$ while the measure dk is dimensionless. This necessitates the factor of $r^{-(n-2)}$ on the right hand side.

Substituting this into (2.12) we find that, up to a finite constant factor independent of the initial and final states,

$$I = \frac{1}{r_1^{n-2} r_2^{n-2}} \int dt d^n x d\lambda \sinh^{n-1} \lambda \delta(r_1^2, r^2) \delta(t_1, t) \delta(r_2^2, r_\lambda^2) \delta(t_2, t_\lambda), \quad (2.15)$$

where the subscript λ indicates that the quantity is boosted in the (x^0, x^1) plane with parameter λ ; that is,

$$\begin{pmatrix} t_\lambda \\ (x^1)_\lambda \end{pmatrix} = \begin{pmatrix} t \cosh \lambda + x^1 \sinh \lambda \\ t \sinh \lambda + x^1 \cosh \lambda \end{pmatrix}, \quad (2.16)$$

and $x_\lambda^i = x^i$ for $i > 1$.

Note that for $n = 1$ the integral I can be easily done. We use three of the δ -functions to integrate over dt , dx and $d\lambda$, obtaining a result that is finite in the distributional sense:

$$I_{n=1} = \delta((x_1^2 - t_1^2), (x_2^2 - t_2^2)). \quad (2.17)$$

This expression is manifestly Lorentz invariant. The convergence for $n = 1$ is not surprising as, in this case, there is only one constraint and it has a well-behaved spectrum (satisfying, for example, property A of [19]). For this kind of system, the averaging procedure converges in the same way that $\int e^{ikx} dx$ converges to $\delta(k)$ as a distribution over C_0^∞ . Here, k plays the role of the spectral parameter of the constraint.

From now on we will consider only the case $n > 1$. Using the (t_1, t) , (t_2, t_λ) and (r_2^2, r_λ^2) delta-functions to do the $d^n x$ and dt integrations in (2.15), we obtain, again up to an overall constant factor,

$$I = \frac{\delta(s_1^2, s_2^2)}{r_1^{n-2} r_2^{n-2}} \int \left[r_1^2 \sinh^2 \lambda - (t_2 - t_1 \cosh \lambda)^2 \right]^{\frac{n-3}{2}} \sinh \lambda d\lambda, \quad (2.18)$$

where $s_a^2 = \eta_{\mu\nu} x_a^\mu x_a^\nu$, $a = 1, 2$ and λ is integrated over all positive values such that the term in square brackets is positive. Changing variables to $\xi = \sinh \lambda$ this can be written,

$$I = \frac{\delta(s_1^2, s_2^2)}{r_1^{n-2} r_2^{n-2}} \int d\xi \left[s_1^2 \xi^2 + 2t_1 t_2 \xi - (r_1^2 + t_2^2) \right]^{\frac{n-3}{2}}. \quad (2.19)$$

It is now convenient to treat independently the cases where s_1 and s_2 are either both spacelike or both timelike. We will not treat the lightlike case, and it is clear from (2.10) that I will vanish if x_1 is timelike while x_2 is spacelike.

1. s_1, s_2 Spacelike

In this case, the term in square brackets in (2.19) will be positive for ξ greater than some ξ_0 . The integral has an infinite domain and will, in general, diverge. Now define the dimensionless parameter,

$$u = \frac{s_1^2 \xi + t_1 t_2}{r_1 r_2} . \quad (2.20)$$

The interval $\xi \in [\xi_0, \infty)$ maps to $u \in [1, \infty)$ while the point $\xi = 1$ maps to $u = u_0$, with

$$u_0 = \frac{s_1^2 + t_1 t_2}{r_1 r_2} . \quad (2.21)$$

In the appendix we show that $u_0 < 1$.

In terms of u , Eq. (2.19) takes the form

$$I = \frac{\delta(s_1^2, s_2^2)}{s_1^{n-1}} \int_1^\infty du (u^2 - 1)^{\frac{n-3}{2}} . \quad (2.22)$$

Hence, we have succeeded in writing I as a divergent factor times a Lorentz invariant quantity:

$$I = \lim_{\Lambda \rightarrow \infty} \frac{\delta(s_1^2, s_2^2)}{s_1^{n-1}} \Delta(\Lambda) , \quad (2.23)$$

where

$$\Delta(\Lambda) = \int_1^\Lambda du (u^2 - 1)^{\frac{n-3}{2}} .$$

This diverges as Λ^{n-2} for $n > 2$ and as $\log(\Lambda)$ for $n = 2$.

2. s_1, s_2 Timelike

In this case, the term under square brackets in (2.19) will be negative for all values of ξ greater than some ξ_0 , which will be the upper limit of the domain of integration. The integral I is therefore convergent. In the present case we define

$$u = -\frac{s_1^2 \xi + t_1 t_2}{r_1 r_2} \quad (2.24)$$

and (2.19) takes the form

$$I = \frac{\delta(s_1^2, s_2^2)}{s_1^{n-1}} \int_1^1 du (1 - u^2)^{\frac{n-3}{2}} , \quad (2.25)$$

where the lower limit of integration will be the maximum of -1 and

$$u_0 = \frac{-(s_1^2 + t_1 t_2)}{r_1 r_2} .$$

As shown in the Appendix, u_0 can be either greater than 1 (when t_1 and t_2 have different sign) or less than -1 (when t_1 and t_2 have the same sign). Thus, as expected, the integral I vanishes when (say) x_1 lies in the future lightcone and x_2 lies in the past. If, on the other hand, t_1 and t_2 have the same sign, then

$$I = \Sigma \frac{\delta(s_1^2, s_2^2)}{s_1^{n-1}}, \quad (2.26)$$

where

$$\Sigma = \int_{-1}^1 du (1 - u^2)^{\frac{n-3}{2}}. \quad (2.27)$$

For $n > 1$, this integral is convergent, and its value is

$$\Sigma = \frac{\sqrt{\pi} \Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}. \quad (2.28)$$

C. A candidate for the rigging map

At this point we have succeeded in regularizing the divergent integrals that arise when averaging distributional states over $SO_c(n, 1)$. Take now $\Phi \subset \mathcal{H}_{\text{aux}}$ to be the set of functions with compact support not intersecting the light cone. It follows from our work above that the averaging procedure converges for states ϕ supported inside the lightcone. Let us now consider the case of x_1, x_2 outside the light cone. Note that, given two such points x_1 and x_2 , the expression (2.20) for u defines a function $u(h)$ for $h \in H_+^n$. To define the physical inner product of states supported outside the lightcone, we will “renormalize” the averaging integrals by dividing by $\Delta(\Lambda)$. Let us define an object Q by the expression:

$$\langle x_1 | Q | x_2 \rangle = \lim_{\Lambda \rightarrow \infty} \frac{\int_{g \in G_\Lambda(x_1, x_2)} \langle x_1 | U(g) | x_2 \rangle dg}{\Delta(\Lambda)}, \quad (2.29)$$

where $G_\Lambda(x_1, x_2)$ is the compact subset of G given by $g = hk$, $k \in K$, $h \in H_+^n$ with h such that $u(h) < \Lambda$. The results of the previous subsection show that this expression converges to a distribution in x_1, x_2 given by

$$\langle x_1 | Q | x_2 \rangle = \frac{1}{|x_1|^{n-1}} \delta(x_1^2, x_2^2). \quad (2.30)$$

for x_1, x_2 outside the light cone.

While this has the same form as the group averaging result (2.26) inside the light cone, we should recall that it is in reality not the same object; the limit (2.29) would vanish for any x_1, x_2 inside the light cone. Thus, we have a domain Φ_1 of functions of compact support inside the light cone and a domain Φ_2 of functions of compact support outside the light cone with $\Phi = \Phi_1 \oplus \Phi_2$. On Φ_1 , we have a rigging map η_1 defined by group averaging. For $\phi_2 \in \Phi_2$, we have a candidate rigging map η_2 defined by

$$\eta_2|\phi_2\rangle = \langle\phi_2|Q, \quad (2.31)$$

where we have established that this expression defines an element of Φ^* , the algebraic dual of Φ , as is appropriate for a rigging map [8].

III. RIGGING MAPS

In section II, we used a ‘renormalization’ procedure to arrive at a candidate rigging map η_2 , for the region outside the light cone: $[\eta_2|x_1\rangle](|x_2\rangle) = |x_1|^{n-1}\delta(x_1^2, x_2^2)$ for $x_1^2, x_2^2 > 0$. This certainly appears to be a reasonable choice (it gives the ‘natural’ inner product on physical states), but we should take care to check that it does indeed fulfill the requirements of refined algebraic quantization. It is clearly real, symmetric, and positive. Thus, the only remaining requirement [8] is that η_2 commute with the action of the observables. For the obvious observables (the invariant distance s^2 from the origin or observables associated with the vector field $\frac{\partial}{\partial s}$) this is again trivial.

However, the definition of observable used in RAQ is rather subtle, so that we cannot be sure that this list is exhaustive. Thus, a proof is required to show that η_2 commutes with the observables. This is given by a computation in subsection A below. We will then show in subsection B that any map of the form $a_1\eta_1 \oplus a_2\eta_2$ (for $a_1, a_2 \in \mathbf{R}^+$) is a rigging map, where η_1 denotes group averaging on states supported inside the light cone. By this notation we mean that, for $\phi_1, \tilde{\phi}_1 \in \Phi_1$ and $\phi_2, \tilde{\phi}_2 \in \Phi_2$,

$$[(a_1\eta_1 \oplus a_2\eta_2)(\phi_1 + \phi_2)](\tilde{\phi}_1 + \tilde{\phi}_2) = a_1[\eta_1\phi_1](\tilde{\phi}_1) + a_2[\eta_2\phi_2](\tilde{\phi}_2). \quad (3.1)$$

The statement that $a_1\eta_1 \oplus a_2\eta_2$ is a rigging map again requires a proof that it commutes with the observables. We proceed by deriving a general result: Given a suitable decomposition $\Phi = \Phi_1 \oplus \Phi_2$ and rigging maps η_1 and η_2 on Φ_1 and Φ_2 separately, the fact that group averaging converges on Φ_1 but not on Φ_2 means that $a_1\eta_1 \oplus a_2\eta_2$ is a rigging map. Along the way, we come to an improved understanding of the interaction between RAQ and superselection rules.

A. η_2 is a rigging map on Φ_2

To show that η_2 is a rigging map on Φ_2 , we must verify that η_2 commutes with the action of observables on Φ_2 . As we will see, the proof is trivial for the groups $SO_c(1, 1)$ and $SO_c(2, 1)$, but a calculation is required for $SO_c(n, 1)$ when n is larger than 2. For $SO_c(1, 1)$, group averaging in fact converges so that the associated η_2 is clearly a rigging map. For $SO_c(2, 1)$, taking the leading order divergence of (2.19) gives a result proportional to our candidate rigging map (2.30). Thus, the cut-off may be imposed in a state-independent manner. It follows that the candidate map may be written $\eta_2 = \lim_{\Lambda \rightarrow \infty} \eta_{2,\Lambda}$, where

$$\eta_{2,\Lambda} = \frac{\int_{K_\Lambda} dg U(g)}{N(K_\Lambda)}, \quad (3.2)$$

for a sequence K_Λ of compact subsets of $SO_c(2, 1)$ given by elements of the form (2.4), (2.6) with $\lambda < \Lambda$ and an appropriate function N . As a result, any observable \mathcal{O} commutes with $\eta_{2,\Lambda}$ for all Λ . Using the fact that each $\phi \in \Phi$ acts continuously on Φ' , we may pass to the limit. It then follows that \mathcal{O} commutes with η_2 .

For $n \geq 3$, the limit by which η_2 is defined is more complicated as we must use the sets $G_\Lambda(x_1, x_2)$ which do in fact depend on the points x_1 and x_2 . Thus, the fact that \mathcal{O} commutes with $U(g)$ no longer guarantees that it commutes with a regularized rigging map. As a result, we need to explicitly verify that η_2 commutes with the action of observables for the cases $n \geq 3$.

It will be convenient to label points outside the light cone with the invariant distance s from the origin and a point θ on the unit hyperboloid $x^2 = +1$. We introduce the distributional states $|s, \theta\rangle = s^{n/2}|x(s, \theta)\rangle$ satisfying $\langle s_1, \theta_1 | s_2, \theta_2 \rangle = \delta(s_1, s_2)\delta(\theta_1, \theta_2)$ where $\int d\theta \delta(\theta, \theta_0) = 1$ for the invariant measure $d\theta$ on the hyperboloid. For any observable $\mathcal{O} : \Phi_2 \rightarrow \Phi_2$, both $\eta_2 \circ \mathcal{O}$ and $\mathcal{O} \circ \eta_2$ define maps from Φ_2 to its algebraic dual, Φ_2^* . Thus, given $\phi, \psi \in \Phi_2$, we have $[\mathcal{O} \circ \eta_2(\phi)](\psi) \in \mathbf{C}$ (where \mathbf{C} denotes the complex numbers), and similarly for $\eta_2 \circ \mathcal{O}$. Thus, the objects $[\mathcal{O} \circ \eta_2(|x_1\rangle)](|x_2\rangle)$ and $[\eta_2 \circ \mathcal{O}(|x_1\rangle)](|x_2\rangle)$ both define distributions on $M^{n,1} \times M^{n,1}$. If these distributions coincide, then η_2 commutes with \mathcal{O} .

In terms of our states $|s, \theta\rangle$, the map η_2 can be written

$$\eta_2|\phi\rangle = \langle\phi|Q = \int ds \left(\int d\theta \langle\phi|s, \theta\rangle \right) \left(\int d\theta \langle s, \theta| \right). \quad (3.3)$$

The distributions are therefore:

$$\begin{aligned} [\mathcal{O}^\dagger \circ \eta_2(|x_1\rangle)](|x_2\rangle) &=: \langle x_1 | Q \mathcal{O} | x_2 \rangle \\ [\eta_2 \circ \mathcal{O}^\dagger(|x_1\rangle)](|x_2\rangle) &=: \langle x_1 | \mathcal{O} Q | x_2 \rangle. \end{aligned} \quad (3.4)$$

Let us denote by $\mathcal{A}_{2,2}$ the set of observables that map Φ_2 to Φ_2 . Using the fact² that \dagger is an involution on $\mathcal{A}_{2,2}$, showing that η_2 commutes with the observables is equivalent to showing $\langle x_1 | Q \mathcal{O} | x_2 \rangle = \langle x_1 | \mathcal{O} Q | x_2 \rangle$ for all $\mathcal{O} \in \mathcal{A}_{2,2}$.

We now begin a computation. Let us pick a reference point θ_0 on the unit hyperboloid $x^2 = +1$ and, for any other point θ on the unit hyperboloid, an $SO_c(n, 1)$ element $g(\theta, \theta_0)$ that moves θ_0 to θ . Also, note that since the measure $d\theta$ is invariant under $SO_c(n, 1)$, we have $U(g)Q = Q = QU(g)$ for any $g \in G$. We may therefore write

$$\begin{aligned} \langle s_1, \theta_1 | Q \mathcal{O} | s_2, \theta_2 \rangle &= \langle s_1, \theta_1 | Q \mathcal{O} U(g(\theta_2, \theta_0)) | s_2, \theta_0 \rangle \\ &= \langle s_1, \theta_1 | Q U(g(\theta_2, \theta_0)) \mathcal{O} | s_2, \theta_0 \rangle \\ &= \int d\theta \langle s_1, \theta | \mathcal{O} | s_2, \theta_0 \rangle, \end{aligned} \quad (3.5)$$

²It is not necessarily true that $A^{\dagger\dagger} = A$ for every $A \in \mathcal{A}_{2,2}$. However, it must be true that the domain of $A^{\dagger\dagger}$ includes Φ_2 , and that A and $A^{\dagger\dagger}$ agree when restricted to Φ_2 . As a result, A and $A^{\dagger\dagger}$ may be identified for our purposes.

where, in the last line, we have absorbed U into Q and used the explicit form (3.3) of Q .

It will be useful now to set $\theta_0 = (0, 1, 0, \dots, 0)$, and split the domain of integration in (3.5) into two regions, F and B , the “front” and the “back” of the unit hyperboloid, defined by the sign of x^1 , *i.e.*, $\theta \in F$ if $x^1 \geq 0$, $\theta \in B$ if $x^1 \leq 0$. Now, given any state $|s, \theta\rangle$ in F we can always write it as $U(\theta, \theta_0)|s, \theta_0\rangle$, where $U(\theta, \theta_0)$ is a Lorentz transformation associated³ with the plane defined by the origin of coordinates and the points θ and θ_0 . Note that if $\theta \in F$, the intersection of this plane with the unit hyperboloid will always define a geodesic passing through θ and θ_0 (there may be a disconnected geodesic as well). The inverse Lorentz transformation $[U(\theta, \theta_0)]^{-1}$ must take θ_0 to a point located symmetrically with respect to θ along this geodesic. We may write this as

$$U^{-1}(\theta, \theta_0)|s, \theta_0\rangle = R_1|s, \theta\rangle, \quad (3.6)$$

where R_1 is the reflection through the x^1 axis. This reflection acts on any point x by changing the sign of each coordinate except x^1 . Similarly, we define the other reflection operators R_μ . The integral in (3.5) now takes the form

$$\int d\theta \langle s_1, \theta | \mathcal{O} | s_2, \theta_0 \rangle = \int_F d\theta \langle s_1, \theta | \mathcal{O} | s_2, \theta_0 \rangle + \int_B d\theta \langle s_1, \theta | \mathcal{O} | s_2, \theta_0 \rangle. \quad (3.7)$$

Since the measure $d\theta$ is invariant under reflections and since R_1 preserves the distinction between front and back, for the integral over F we have

$$\begin{aligned} \int_F d\theta \langle s_1, \theta | \mathcal{O} | s_2, \theta_0 \rangle &= \int_F d\theta \langle s_1, \theta | R_1 \mathcal{O} | s_2, \theta_0 \rangle = \int_F d\theta \langle s_1, \theta_0 | U(\theta, \theta_0) \mathcal{O} | s_2, \theta_0 \rangle \\ &= \int_F d\theta \langle s_1, \theta_0 | \mathcal{O} U(\theta, \theta_0) | s_2, \theta_0 \rangle = \int_F d\theta_0 \langle s_1, \theta_0 | \mathcal{O} | s_2, \theta \rangle, \end{aligned}$$

where we have used (3.6), the fact that $U(\theta, \theta_0)$ commutes with \mathcal{O} , and the definition of $U(\theta, \theta_0)$. For the integral over B we first note the following identities:

$$U(\theta, \theta_0) J_{12}(\pi) |\theta_0\rangle = I |\theta\rangle \quad (3.8)$$

$$J_{12}(\pi) U^{-1}(\theta, \theta_0) |\theta_0\rangle = R_2 |\theta\rangle, \quad (3.9)$$

where I is a reflection through the origin, changing the sign of all coordinates and therefore exchanging front and back. The symbol $J_{12}(\pi)$ denotes a rotation by π in the (x^1, x^2) -plane. In this case we have,

$$\begin{aligned} \int_B d\theta \langle s_1, \theta | \mathcal{O} | s_2, \theta_0 \rangle &= \int_F d\theta \langle s_1, \theta | I \mathcal{O} | s_2, \theta_0 \rangle = \int_F d\theta \langle s_1, \theta_0 | J_{12}(\pi) U^{-1}(\theta, \theta_0) \mathcal{O} | s_2, \theta_0 \rangle \\ &= \int_F d\theta_0 \langle s_1, \theta | \mathcal{O} R_2 | s_2, \theta_0 \rangle = \int_B d\theta \langle s_1, \theta_0 | \mathcal{O} | s_2, \theta \rangle. \end{aligned}$$

It follows that we have $\int d\theta \langle s_1, \theta | \mathcal{O} | s_2, \theta_0 \rangle = \int d\theta \langle s_1, \theta_0 | \mathcal{O} | s_2, \theta \rangle$ and $Q\mathcal{O} = \mathcal{O}Q$. Thus, we have shown that η_2 commutes with any observable $\mathcal{O} \in \mathcal{A}_{2,2}$.

³The area element of a plane picks out a 2-form, and therefore the generator of a one-parameter subgroup of the Lorentz group.

B. A Superselection Rule

Here, we wish to show that $a_1\eta_1 \oplus a_2\eta_2$ is a rigging map on $\Phi_1 \oplus \Phi_2$, where Φ_1 is the space of smooth functions supported on compact sets *inside* the light cone. Again, the main issue is to show that our putative rigging map commutes with the relevant set of observables.

Let us refer to the Hilbert space associated with functions supported inside the light cone as \mathcal{H}_1 , and that associated with functions outside the light cone as \mathcal{H}_2 , so that we have $\mathcal{H}_{\text{aux}} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then, since the associated projectors P_1 and P_2 are observables, we may use them to split the algebra \mathcal{A} of observables into four linear spaces: $\mathcal{A} = \bigoplus_{i,j \in \{1,2\}} \mathcal{A}_{i,j}$, where $A \in \mathcal{A}_{i,j}$ maps Φ_i into Φ_j . The observables in $\mathcal{A}_{1,1}$ need only commute with η_1 . But, η_1 is given by convergent group averaging, so this is satisfied. Similarly, observables in $\mathcal{A}_{2,2}$ need only commute with η_2 , and this was checked in subsection A. Thus, we need only consider the observables in $\mathcal{A}_{1,2}$ and $\mathcal{A}_{2,1}$.

Lest the reader think that $\mathcal{A}_{1,2}$ and $\mathcal{A}_{2,1}$ are clearly empty and the result is trivial, we recall from [13] that since group averaging converges both inside and outside the light cone for $SO_c(1,1)$, a nontrivial element of $\mathcal{A}_{1,2}$ in that case is given by the expression

$$\int dg U(g)|\phi_2\rangle\langle\phi_1|U(g^{-1}) \quad (3.10)$$

for any $\phi_1 \in \Phi_1$ and any $\phi_2 \in \Phi_2$. For the case of $SO_c(n,1)$ with $n > 1$, it is unclear whether $\mathcal{A}_{1,2}$ is in fact empty, but in any case our proof below is sufficient.

We begin with a Lemma.

Lemma. *Suppose that we have*

- 1) *a unitary representation of a group G on a Hilbert space \mathcal{H}_{aux} ,*
- 2) *a decomposition $\mathcal{H}_{\text{aux}} = \mathcal{H}_1 \oplus \mathcal{H}_2$ that reduces the group action; that is, for which both \mathcal{H}_1 and \mathcal{H}_2 are invariant under the group action, and*
- 3) *a dense subspace Φ of \mathcal{H}_{aux} whose intersection Φ_1 with \mathcal{H}_1 has the property that, for all $\phi_1, \phi_1' \in \Phi_1$, the matrix elements $\langle\phi_1|U(g)|\phi_1'\rangle$ define an L^1 function on the group G with respect to some measure dg on G .*

Let us denote the intersection $\Phi \cap \mathcal{H}_2$ by Φ_2 and define $\mathcal{A}_{i,j}$ as above. In this case, for any state ϕ_2 of the form $\mathcal{O}\phi_1$ for $\mathcal{O} \in \mathcal{A}_{2,1}$ and $\phi_1 \in \Phi_1$, the matrix elements $\langle\phi_2|U(g)|\phi_2\rangle$ are also L^1 with respect to dg .

Proof. To see this, we simply choose such $\mathcal{O}, \phi_1, \phi_2$. We have

$$\langle\phi_2|U(g)|\phi_2\rangle = \langle\phi_1|U(g)\mathcal{O}^\dagger\mathcal{O}|\phi_1\rangle. \quad (3.11)$$

Since $\mathcal{O}^\dagger\mathcal{O}$ maps Φ_1 to Φ_1 , these matrix elements define an L^1 function on G . \square

Note that the measure to which this Lemma refers need not be the one associated with group averaging. In this way, the Lemma shows that if the fall-off rate of $\langle\phi_1|U(g)|\phi_1\rangle$ can be bounded in some uniform way on Φ_1 , then this same bound also applies to $\mathcal{O}\phi_1$. Clearly, any other property of these matrix elements on Φ_1 also carries over to $\mathcal{O}\phi_1$.

We are now in a position to prove our main result:

Theorem. Suppose that conditions (1-3) of the above Lemma hold with respect to the measure for group averaging and let η_1 denote the group averaging rigging map on Φ_1 . Suppose also that

4) Given states $\phi_2, \phi'_2 \in \Phi_2$ such that $f(g) := \langle \phi_2 | U(g) | \phi'_2 \rangle$ is L^1 with respect to the group averaging measure, the group average of this quantity is zero.

5) There is a rigging map η_2 on Φ_2 which annihilates all states ϕ_2 in Φ_2 for which $\langle \phi_2 | U(g) | \phi_2 \rangle$ is L^1 with respect to the group averaging measure.

Then, for any $a_1, a_2 \in \mathbf{R}$, the map $a_1\eta_1 \oplus a_2\eta_2$ is a rigging map.

Conditions (4) and (5) may seem a bit awkward. However, they are much easier to verify in practice than the (cleaner) condition that Φ_2 contains no non-trivial L^1 states. In particular, for our choices of $\Phi_1, \Phi_2 \subset L^2(M^{n,1})$, the results of section II show that our case of $G = SO_c(n, 1)$ (for $n > 1$) satisfies the assumptions of this theorem. This follows since group averaging clearly diverges for any state $|\phi\rangle = \int ds d\theta \phi(s, \theta) |s, \theta\rangle \in \Phi_2$, except perhaps when the integral $\int d\theta \phi(s, \theta)$ vanishes for all s . However, in this case $\eta_2|\phi\rangle = 0$.

Proof. It is clear that $\eta = a_1\eta_1 \oplus a_2\eta_2$ commutes with the action of $\mathcal{A}_{1,1}$ and $\mathcal{A}_{2,2}$. Thus, we need only consider the operators in $\mathcal{A}_{1,2}$ and their adjoints in $\mathcal{A}_{2,1}$. So, let $\mathcal{O} : \Phi_1 \rightarrow \Phi_2$ and $\mathcal{O}^\dagger : \Phi_2 \rightarrow \Phi_1$. Recalling that \dagger defines a bijection between $\mathcal{A}_{1,2}$ and $\mathcal{A}_{2,1}$ (see footnote 3), the map η will be a rigging map iff

$$[\eta_2\mathcal{O}\phi_1](\phi_2) = [\eta_1\phi_1](\mathcal{O}^\dagger\phi_2). \quad (3.12)$$

Now, our Lemma tells us that $\mathcal{O}\phi_1$ is an L^1 state in Φ_2 . Thus, by condition (5), η_2 annihilates this state and the left-hand side of (3.12) vanishes. The right-hand side is given by group averaging:

$$[\eta_1\phi_1](\mathcal{O}^\dagger\phi_2) = \int_G dg \langle \phi_1 | U(g) \mathcal{O}^\dagger | \phi_2 \rangle. \quad (3.13)$$

Note that the function $\langle \phi_1 | U(g) \mathcal{O}^\dagger | \phi_2 \rangle$ is L^1 since $\mathcal{O}^\dagger | \phi_2 \rangle \in \Phi_1$. Since $\mathcal{O} | \phi_1 \rangle \in \Phi_2$, expression (3.13) vanishes by condition (4) and we are done. \square

Note that while we were unable to decide whether $\mathcal{A}_{2,1}$ was empty (and thus whether \mathcal{H}_1 and \mathcal{H}_2 are superselected in \mathcal{H}_{aux}), we have shown that any \mathcal{O} in $\mathcal{A}_{1,2}$ acts as the zero operator on the physical Hilbert space so that a superselection rule must exist at the physical level. It is clear that, whether or not a superselection rule exists in \mathcal{H}_{aux} , the ambiguity in the choice of rigging map directly corresponds to superselection rules on the physical phase space⁴.

IV. DISCUSSION

In the above work, we considered a particular regularization of the rigging map given by choosing compact subsets of the gauge group. While we were able to ‘renormalize’ our group

⁴See, however, [20] for subtleties that may arise when further constraints are imposed.

averaging map, the limiting procedure (2.29) defining the physical inner product $[\eta_2(\phi_2)](\phi'_2)$ depended on the states ϕ_2, ϕ'_2 in a rather complicated way. This necessitated the separate proof in section IIIA that our limit did in fact define a rigging map for the case $SO_c(n, 1)$ with $n > 2$, and is not particularly encouraging for the development of a general algorithm. One might expect similar results from other renormalization procedures (such as the one suggested in [21]) which are not manifestly symmetric under the G action⁵.

However, suppose for the moment that we had used a state-independent renormalization scheme to define a map $\alpha : \Phi \rightarrow \Phi^*$ of the form: $\alpha = \lim_{\Lambda \rightarrow \infty} \alpha_\Lambda$ where

$$[\alpha_\Lambda(\phi)](\phi') = N(\Lambda) \int_{G_\Lambda} dg \langle \phi | U(g) | \phi' \rangle, \quad (4.1)$$

with $G_\Lambda \subset G$ containing those elements of the form $\{kb(\lambda)k'\}$ for $k, k' \in K$, $b(\lambda)$ a boost of magnitude λ in the $0, 1$ plane, and $\lambda < \Lambda$. We may take $N(\Lambda)$ to be defined such that $[\alpha_\Lambda(\phi_0)](\phi_0) = 1$ for some reference state ϕ_0 . This leads to imposing a cutoff in terms of the integration variable ξ of (2.19) instead of in terms of u . As one can see from (2.19) and (2.30), this new map can be related to the rigging map η_2 by

$$[\alpha(|x_1\rangle)](|x_2\rangle) = \frac{s_1^{2(n-2)}}{r_1^{n-2}r_2^{n-2}}[\eta_2(|x_1\rangle)](|x_2\rangle), \quad (4.2)$$

for x_1, x_2 outside the light cone. The map α is not a rigging map as it does not solve the constraints. This is evident from the lack of Lorentz invariance in (4.2). Note, however, that since states $\phi_2 \in \Phi_2$ are associated with functions supported on compact sets outside the light cone, the coefficient $\frac{s_1^{2(n-2)}}{r_1^{n-2}r_2^{n-2}}$ is strictly positive and is bounded on any such compact set. As a result, if we restrict attention to the action of α and η_2 on positive functions, the maps α and η_2 have the same domain and the same kernel. In general, a study of the maps (4.1) for various choices of $N(\lambda)$ may lead to a detailed knowledge of superselection sectors, as we now discuss.

Suppose that we have $\Phi = \Phi_1 \oplus \Phi_2$ and that the two spaces are in some sense characterized by different rates of divergence of the limit (4.1), say with the integral diverging faster on Φ_2 than Φ_1 . One might expect that through suitable renormalization one can define rigging maps η_1 and η_2 on Φ_1 and Φ_2 , with η_2 requiring a stronger renormalization than η_1 . In analogy with the Lemma of the last subsection, we expect the action of η_1 can be defined on the image of Φ_1 under any observable. We also expect a parallel with the subsequent theorem. Let us replace assumptions (4) and (5) with:

4') Given states $\phi_2, \phi'_2 \in \Phi_2$ such the limit defining $[\eta_1(\phi_2)](\phi'_2)$ converges, the limit of this quantity is zero.

5') There is a rigging map η_2 on Φ_2 which annihilates all states ϕ_2 in Φ_2 for which the limit defining $[\eta_1(\phi_2)](\phi_2)$ converges.

⁵It might be of interest to determine if the scheme of [21] requires ‘state-dependent regularization’ in the case where group averaging converges.

Since the map η_2 involves a stronger renormalization than η_1 , we may expect property (5') to hold. On the other hand, one might arrange for property 4' to hold by simply assigning to Φ_1 any state ϕ_2 for which the limit defining $\eta_1(\phi_2)(\phi_2)$ converges to a nonzero value. Under these conditions, the argument proceeds in exact parallel with section IIIB. We conclude that $a_1\eta_1 \oplus a_2\eta_2$ is a rigging map, and that the images of η_1 and η_2 are superselected in the physical Hilbert space. In this way, it may be generally true that spaces of functions for which the group averaging integral diverges at different rates are superselected in the physical Hilbert space.

However, certain subtleties remain to be explored. For example, let us return for a moment to the case of $SO_c(n, 1)$ acting on $L^2(M^{n,1})$. There are of course functions supported *inside* the light cone for which group averaging does not converge. These are simply functions whose support is not compact. Thus, one might conceivably attempt to renormalize the group averaging map on a space of such functions associated with the interior of the light cone. In this case, it is *not* clear that a physical superselection rule results. This issue, and others, we leave for future investigation.

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APPENDIX A:

a. Spacelike Case

For $s_1=s_2$ spacelike, we parameterize

$$\begin{aligned} r_i &= s \cosh \tau_i \\ t_i &= s \sinh \tau_i , \end{aligned}$$

where $i = 1, 2$, s is a positive number and $\tau_i \in (-\infty, \infty)$. We start with the identity

$$-\cosh(\tau_1 + \tau_2) \leq 1 \leq \cosh(\tau_1 - \tau_2) ,$$

which can be rewritten,

$$-1 \leq \frac{1 + \sinh \tau_1 \sinh \tau_2}{\cosh \tau_1 \cosh \tau_2} \leq 1 .$$

This last inequality tells us that

$$-1 \leq \frac{s_1^2 + t_1 t_2}{r_1 r_2} \leq 1 .$$

b. Timelike Case

For $s_1=s_2$ timelike, we parameterize

$$\begin{aligned} t_i &= \beta_i s \cosh \tau_i \\ r_i &= s \sinh \tau_i , \end{aligned}$$

where now τ_i is restricted to the interval $[0, \infty)$, and β_i are ± 1 and set the sign of t_i . Therefore

$$\frac{s_1^2 + t_1 t_2}{r_1 r_2} = \frac{\alpha \cosh \tau_1 \cosh \tau_2 - 1}{\sinh \tau_1 \sinh \tau_2} ,$$

where $\alpha = \beta_1 \beta_2$. Consider $\alpha = 1$. The identity

$$\cosh(\tau_1 + \tau_2) \geq 0$$

can be written

$$\frac{\cosh \tau_1 \cosh \tau_2 - 1}{\sinh \tau_1 \sinh \tau_2} \geq 1 ,$$

which shows that

$$\frac{s_1^2 + t_1 t_2}{r_1 r_2} \geq 1 .$$

For $\alpha = -1$ we check analogously that

$$\frac{s_1^2 + t_1 t_2}{r_1 r_2} \leq -1 .$$

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